# Solution to Class Exercise 8

Let

$$\mathbf{F} = \frac{1}{x^2+y^2}(x\mathbf{i}+y\mathbf{j}) \ .$$

1. Find the work done under $\mathbf{F}$ from $(1,0)$ to $(3,0)$ along the $x$-axis and then along upper half circle arc $(x-2)^2+y^2=1$.

   The straight line along the $x$-axis is given by $\gamma(t) = (2t+1)\mathbf{i}$ , $\quad t \in [0,1]$ . Therefore, the work done along this line is

   $$\int_0^1 \frac{2t+1}{(2t+1)^2+0^2} \times 2\,dt = 2\int_0^1 \frac{1}{2t+1}\,dt = \log 3 \ .$$

   The half circle is given by $\mathbf{r}(\theta) = (2+\cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$ , $\theta \in [0,\pi]$ . Be careful, $\mathbf{r}(0) = (3,0)$ and $\mathbf{r}(\pi) = (1,0)$ so the orientation is reversed. Therefore, the work done is

   $$\begin{aligned} & -\int_0^\pi \frac{(2+\cos\theta)\mathbf{i}+\sin\theta\mathbf{j}}{(2+\cos\theta)^2+\sin^2\theta} \cdot (-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\,d\theta \\ = & -\int_0^\pi \frac{-2\sin\theta}{5+4\cos\theta}\,d\theta \\ = & -2\int_1^{-1} \frac{dt}{5+4t} \\ = & \log 3 \ . \end{aligned}$$

2. Is $\mathbf{F}$ conservative in $\mathbb{R}^2 \setminus \{(0,0)\}$?

   The potential is given by $\frac{1}{2}\log(x^2+y^2)$ . Hence no matter which path you take to go from $(1,0)$ to $(3,0)$ the work done is the same

   $$\int_{(1,0)}^{(3,0)} \mathbf{F}\cdot\, d\mathbf{r} = \frac{1}{2}\log 9 - \frac{1}{2}\log 1 = \log 3 \ .$$